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EQUATIONS OF MOTION FOR THE AIR-LUBRICATED FLEXIBLE DISC

by

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ABSTRACT: Equations are formulated for the motion of a spinning elastic disc over an air bearing. Some approximation methods are indicated, but no detailed solutions are carried out.

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INTRODUCTION

For use in random-access memory units, a rotating flexible-disc file has certain advantages over the more traditional rigid-disc file. The most obvious advantages are lighter weight and less safety hazard. A somewhat more subtle, but extremely important, advantage is a direct consequence of the flexibility: in a rigid-disc file, the read-write circuitry is carried in a spring-cantilever mounted slider bearing, and radial accessing to the various tracks is accomplished by a servo-system; a flexible-disc file can accommodate to a head-bar, in which the circuitry for all recording tracks can be carried, thus reducing the radial access time to zero. Run-out precludes the use of a head-bar with a rigid disc.

The spacing between the discs and the head-bar is maintained by a lubricating film of air. The dynamics of the air-lubricated flexible disc are considerably more complicated than those of the rigid-disc file, for the lubrication equation is coupled to the equations of a spinning elastic plate.

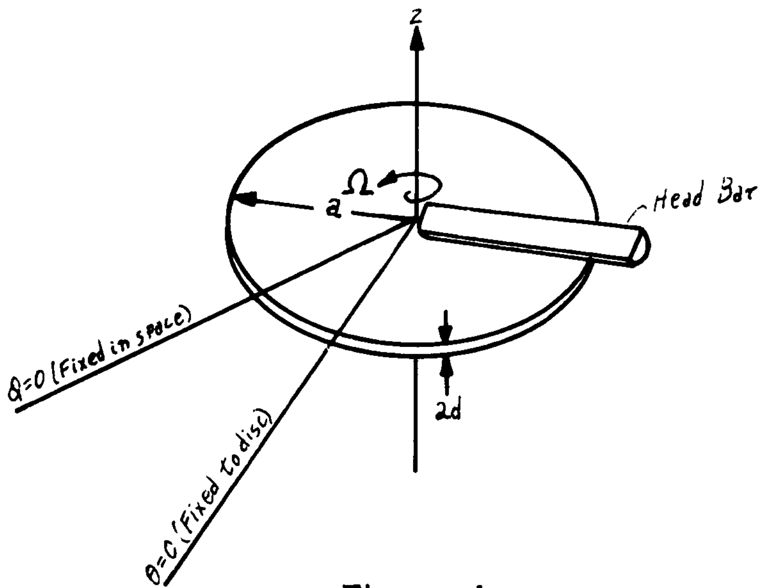

Figure 1

The unsteady motion of elastic discs has been extensively studied. The early work of Kirchoff [1,2] in 1850 provided deflexion shapes and frequencies of linear transverse flexural vibrations of non-rotating circular plates (there are many modern accounts of this[3]). Since then many writers have contributed to the theory of vibrations of non-rotating plates, both for asymmetrical and axis-symmetrical modes.

For the centrally supported, rotating circular disc, contributions to the literature are neither numerous nor complete. The importance of flexural wave motions in such discs was recognized as early as 1872 by Richards,[4] who reported, in reference to circular sawblades:

> For a time, and up to a certain point, the rigidity or stiffness of the plate will increase; after this it begins to diminish, until, at a very high velocity, it becomes as limber and pliant as a piece of paper, and finally will, on its periphery, assume a series of undulations or waves, and is as sensitive to pressure, on the side of the plate, as though it were of paper.

Subsequent experimental investigations by Campbell[5] in 1925 and von Freudenreich[6] clearly established the stationary wave (a flexural wave which travels opposite to the direction of rotation of the disc at a speed equal to the speed of rotation, so that, to a stationary observer,it appears fixed)as a significant cause of rotating disc failures. Still later, in 1957-8, Tobias and Arnold[7,8] qualitatively studied the effect of imperfection on flexural waves and vibrations in spinning discs, particularly in the nonlinear range.

The earliest theory of spinning disc vibrations is due to Lamb and Southwell,[9] who used Rayleigh's method to obtain upper and lower bounds for the natural frequencies of a disc, whole at its center. In formulating their results they obtained an exact solution for the vibrations of a spinning membrane in terms of hypergeometrical series. In a subsequent paper Southwell[10] discussed the effect of a finite central clamping area in the spinning disc. For the spinning membrane, Southwell found that the presence of central clamping alters the shape of axisymmetric vibrations but not the frequency. However, Simmonds[18] has pointed out that this conclusion is incorrect.

In the use of Rayleigh's method, a radial mode shape is assumed. Southwell[10] notes that the nearness of the upper and lower frequency bounds is very sensitive to the choice of radial modal shape - and that, through improper choice, these bounds may differ by as much as 100%. A comparison of calculated upper-bound frequencies[9] with experimental data[8] was made by Bulkeley,[11] for asymmetric disc vibrations having 2, 3, and 4 nodal diameters. Agreement is good for low disc speeds in the mode having 2 nodal diameters. Agreement is not good for increasing disc speeds and for higher modes. The value calculated for the disc speeds at which the gravest stationary wave (two nodal diameters) occurs differs from the experimental value by greater than 3 per cent - and is much worse for stationary waves having a greater number of nodal diameters.

A. Stodola[12] gave a Rayleigh-Ritz procedure for calculating the frequencies of spinning turbine discs. His work is different from that of Lamb and Southwell[9,10] in that his analysis allows for discs of variable thickness (radially) and assumes a different form for the radial mode shape. No comparison of this analysis with experiments is reported.

Asymmetrical nonlinear vibrations of non-rotating discs were studied by Tobias[13] in 1957, with particular attention being given to a qualitative description of the effects of disc imperfection. This work uses the approximate technique of Lamb and Southwell[9] in an attempt to extend the results of Zenneck,[14] who discussed the effect of slight imperfection on linear disc vibrations. Tobias developed Lagrangian equations of motion for the disc, assuming a form for the deflected shape. These equations are nonlinear, middle surface stretching terms being included in the disc strain energy. Points in the disc are assumed to move only perpendicular to its plane. This violates boundary conditions and equilibrium in the plane of the disc, but in approximate analyses of this type some boundary conditions can be ignored.[15,16] To neglect equilibrium in the plane of the disc is equivalent to constraining its motion, making it "stiffer" than it actually is.

With Tobias's assumptions about displacements, it is a simple matter to extend his results for the non-rotating disc to the case where the disc spins.[11] A simple relation between the speed of disc rotation and the amplitude of stationary waves present is obtained.

No theory of spinning disc vibrations and waves includes the effects of transverse shear or rotatory inertia.

The stationary wave phenomenon has been found to be of great practical importance in spinning, centrally supported, shallow shell segments,[17] as it is the cause of many shell failures (as in spinning discs). There is, however, no theory available which predicts the shape of stationary waves in spinning shells, or at what speeds of rotation they may occur.

In this report, we set out the equations governing the motion of the flexible disc file. Some approximation methods are suggested, but no solutions are carried out.

I. THE SPINNING-DISC EQUATIONS

The development presented here for the equations of disc motion is originally due to von Kármán[19]; their derivation in rectangular coordinates is given by Timoshenko and Woinowsky-Krieger[20]. The equations are nonlinear because middle-surface stretching in the disc is taken into account. In calculating the strain components, we retain the linear and quadratic terms in the displacement gradients. We assume the disc to be perfectly circular, uniform, homogeneous, isotropic, and Hookean.

Figure 2 shows a typical element of a circular disc, together with middle-surface stretching forces per unit length N_r, N_θ, $N_{r\theta}$, $N_{\theta r}$, acting on its faces. We obtain equations of motion for the disc by

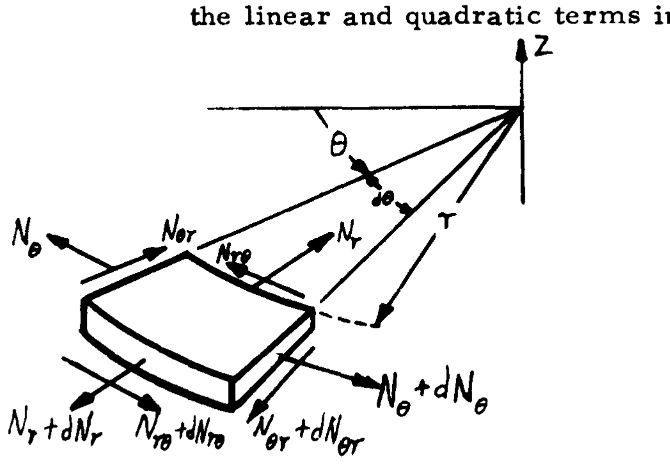

Figure 2

writing expressions of 1) Newton's second law, 2) compatibility of strain, and 3) Hooke's law for this small element.

Points within the disc may have radial, tangential, and axial components of displacement; we denote these by u, v, w, in the r, Θ, z directions respectively. In writing Newton's second law for the disc element, we neglect the in-plane components of acceleration $\partial^2 u/\partial t^2$ and $\partial^2 v/\partial t^2$; the typical period of in-plane motion is short compared with that of transverse motion, so that effects of the in-plane acceleration on the transverse motion will be averaged out. Thus, the in-plane motion, which is regarded as forced by the transverse motion, is assumed quasi-statis, i.e., at any given time it is governed by the equations of static equilibrium.*

For equilibrium of moments about an axis perpendicular to the plate element of Figure 2, we need

$$N_{r\Theta} = N_{\Theta r} \ . \tag{1.1}$$

Equilibrium in the radial direction requires that

$$r\partial N_r/\partial r + \partial N_{r\Theta}/\partial\Theta + N_r - N_\Theta + rR = 0\,, \tag{1.2}$$

where R denotes the radial body force per unit area of disc; it may be interpreted as a centrifugal force when the spinning disc is treated from the D'Alembert point of view.

For equilibrium in the tangential direction,

$$\partial N_\Theta/\partial\Theta + r\partial N_{r\Theta}/\partial r + 2N_{r\Theta} = 0\ . \tag{1.3}$$

Equations (1.2) and (1.3) govern the plane stress in the disc. However, superimposed on this is the stress field induced by disc flexure.

If stretching forces in the plane of the disc could be ignored, the static transverse deflexion w would satisfy the forced biharmonic equation**

$$\Delta^4 w = (p - p_a)/D\,, \tag{1.4}$$

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* See reference 21, page 56
**See reference 22, page 39-24

where p is the loading pressure, p_a is the ambient pressure,

$$\nabla^2 = \partial^2/\partial r^2 + (1/r)\, \partial/\partial r + (1/r^2)\, \partial^2/\partial \Theta^2, \tag{1.5}$$

and D is the flexural rigidity, defined by

$$D = 2Ed^3/3\ (1-\nu^2), \tag{1.6}$$

with E as Young's modulus and ν Poisson's ratio for the material of the disc.

The angular position of any point in the disc with respect to a fixed observer is

$$\phi = \Theta + \Omega t, \tag{1.7}$$

where Ω is the angular velocity of the disc.

The derivation of equation (1.4) assumes that lines normal to the middle surface of the undeflected disc will remain normal to the middle surface at all times; this is equivalent to neglecting transverse shearing deformation. It also assumes that the disc curvature at any point is small compared with the disc thickness and that the slope of the deflected surface is small compared with unity, i.e., that strains are small compared with unity. These assumptions do not in themselves restrict the magnitude of the disc deflexion, for large-scale rigid-body motions of each disc element may take place even though the strains are small, especially if the disc is very thin.

When stretching forces in the middle surface of the disc must be accounted for, equation (1.4) does not apply. Moreover, the derivation of (1.4) assumes static deflexion. Under certain conditions, time-dependent motion can be assumed quasi-static, so that (1.4) provides an approximation. In general, however, the transverse inertia of the disc cannot be ignored. When stretching forces and transverse inertia are included in the analysis, we obtain instead*

$$D\nabla^4 w = -2\rho d \frac{\partial^2 w}{\partial t^2} + \frac{1}{r^2}\left(N_\Theta \frac{\partial^2 w}{\partial \Theta^2} + \frac{\partial N_\Theta}{\partial \Theta} \frac{\partial w}{\partial \Theta}\right)$$

$$+ N_r \frac{\partial^2 w}{\partial r^2} + \frac{\partial N_r}{\partial r} \frac{\partial w}{\partial r} + \frac{N_r}{r} \frac{\partial w}{\partial r}$$

$$+ \frac{2N_{r\Theta}}{r} \frac{\partial^2 w}{\partial r \partial \Theta} + \frac{1}{r} \frac{\partial N_{r\Theta}}{\partial \Theta} \frac{\partial w}{\partial r} + \frac{1}{r} \frac{\partial N_{r\Theta}}{\partial r} \frac{\partial w}{\partial \Theta} + p - p_a \,. \qquad (1.8)$$

This result can be simplified. By using the in-plane equilibrium equations (1.2) and (1.3), we obtain

$$D\nabla^4 w = -2\rho d \frac{\partial^2 w}{\partial t^2} + 2N_{r\Theta} \frac{\partial}{\partial r}\left(\frac{1}{r} \frac{\partial w}{\partial \Theta}\right) + \left(\frac{N_\Theta}{r} - R\right) \frac{\partial w}{\partial r}$$

$$+ N_r \frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2} N_\Theta \frac{\partial^2 w}{\partial \Theta^2} + p - p_a \,. \qquad (1.9)$$

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* The derivation, using rectangular coordinates, is given in Reference 20, page 378.

If the transverse load p is regarded as known, equations (1.2), (1.3), and (1.9) provide three equations among the quantities N_r, N_Θ, $N_{r\Theta}$, w. A fourth relation, compatibility of stress, is obtained by combining the strain-displacement relations with Hooke's law.

If second order displacement terms are retained, it has been shown[13] that the strains in the disc are given by

$$\epsilon_r = \frac{\partial u}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 , \tag{1.10}$$

$$\epsilon_\Theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \Theta} + \frac{1}{2r^2}\left(\frac{\partial w}{\partial \Theta}\right)^2, \tag{1.10}$$

$$\gamma_{r\Theta} = \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial \Theta} + \frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial \Theta} . \tag{1.12}$$

Hooke's law, written in terms of the stretching forces, is

$$\epsilon_r = \left(N_r - \nu N_\Theta\right) / 2Ed , \tag{1.13}$$

$$\epsilon_\Theta = \left(N_\Theta - \nu N_r\right) / 2Ed , \tag{1.14}$$

$$\gamma_{r\Theta} = N_{r\Theta} / 2Gd , \tag{1.15}$$

where G is the shear modulus, defined by

$$G = E / 2(1+\nu) . \tag{1.16}$$

From equations (1.10), (1.11), (1.12), we obtain an equation of compatibility of strain:

$$\frac{\partial^2}{\partial r\,\partial\Theta}\left(r\gamma_{r\Theta}\right) - \frac{\partial^2\epsilon_r}{\partial\Theta^2} - r\frac{\partial^2}{\partial r^2}\left(r\epsilon_\Theta\right) + r\frac{\partial\epsilon_r}{\partial r} = \frac{\partial^2}{\partial r\,\partial\Theta}\left(\frac{\partial w}{\partial r}\frac{\partial w}{\partial\Theta}\right)$$

$$+\frac{r}{2}\frac{\partial}{\partial r}\left(\frac{\partial w}{\partial r}\right)^2 - \frac{1}{2}\frac{\partial^2}{\partial\Theta^2}\left(\frac{\partial w}{\partial r}\right)^2 - \frac{r}{2}\frac{\partial^2}{\partial r^2}\left[\frac{1}{r}\left(\frac{\partial w}{\partial\Theta}\right)^2\right]. \tag{1.17}$$

With the strains given by equations (1.13), (1.14), (1.15), we obtain the equation of compatibility of stress:

$$2(1+\nu)\frac{\partial^2}{\partial r\partial\Theta}\left(rN_{r\Theta}\right) + \nu\frac{\partial^2 N_\Theta}{\partial\Theta^2} - \frac{\partial^2 N_r}{\partial\Theta^2} - r\frac{\partial^2}{\partial r^2}\left(rN_\Theta\right)$$

$$+\nu r\frac{\partial^2}{\partial r^2}\left(rN_r\right) + r\frac{\partial N_r}{\partial r} - \nu r\frac{\partial N_\Theta}{\partial r} = 2Ed\left\{\frac{\partial^2}{\partial r\partial\Theta}\left(\frac{\partial w}{\partial r}\frac{\partial w}{\partial\Theta}\right)\right.$$

$$\left.+\frac{r}{2}\frac{\partial}{\partial r}\left(\frac{\partial w}{\partial r}\right)^2 - \frac{1}{2}\frac{\partial^2}{\partial\Theta^2}\left(\frac{\partial w}{\partial r}\right)^2 - \frac{r}{2}\frac{\partial^2}{\partial r^2}\left[\frac{1}{r}\left(\frac{\partial w}{\partial\Theta}\right)^2\right]\right\}. \tag{1.18}$$

Equations (1.2), (1.3), (1.9) and (1.18) provide a complete formulation of the relationship between stretching forces and transverse deflexion in a spinning disc. The body force appearing in (1.9) is the centrifugal D'Alembert force, i.e.,

$$R = 2\rho d\,\Omega^2 r\ . \tag{1.19}$$

The loading pressure p is taken as the lubrication pressure arising from the passage of the disc close to the head bar. Thus, p vanishes except near the bar, for ambient pressure obtains on both sides of the disc.

In setting out the lubrication equation, we assume that the only significant in-plane component of disc velocity is the rigid-body motion Ωr in the increasing Θ direction. Specializing the general result obtained by Langlois[23], we then find that, for an isothermal gas,

$$\frac{\partial}{\partial r}\left(r h^3 p\frac{\partial p}{\partial r}\right) + \frac{\partial}{\partial\phi}\left(h^3 p\frac{\partial p}{\partial\phi}\right) = 6\mu\,\Omega r\frac{\partial}{\partial\phi}(hp) + 12\mu r\frac{\partial}{\partial t}(ph)\ , \tag{1.20}$$

where h denotes the spacing between disc and head bar, and μ is the gas viscosity. On the periphery of the lubricating film, the pressure is ambient. The use of Θ as the angular variable in the elastic equations and ϕ in the lubrication equation presents no real difficulty. With equation (1.7),

$$\frac{\partial}{\partial \phi} = \frac{\partial}{\partial \Theta} \quad . \tag{1.21}$$

Consequently, all Θ-derivatives could be replaced by ϕ-derivatives in the elastic equations (not in the lubrication equation, for additional velocity terms would be needed to achieve the transformation from a fixed to a moving coordinate system). Thus, we consider, in essence, a non-rotating disc subjected to two ficticious force fields: 1) a body force field given by (1.19); 2) a transverse loading, determined by (1.20) subject to the boundary condition of ambient pressure on the bearing periphery.

II. BOUNDARY CONDITIONS AT THE DISC PERIPHERY

Boundary conditions for the elastic disc are of two types: those associated with disc flexure; those associated with extension of the middle-surface. Flexure boundary conditions can determine values of deflexion, slope, bending moment, twisting moment, and transverse reaction force at a disc edge; two such conditions may be specified along any segment of the disc periphery. Boundary conditions associated with middle-surface extension can determine values of the stretching forces and in-plane displacements at disc boundaries. We can, in general, specify values of the radial and tangential stretching forces, or we may specify the value of the radial displacement.

The selection of appropriate boundary conditions depends upon the particular problem to be solved. Of particular interest is a disc which is completely free at its outer edge, $r = a$, and is built-in at an inner radius, $r = b$. For such a disc, the boundary conditions are*:

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* See reference 20, page 284

1) At the outer edge:

a) vanishing bending moment

$$D\left[\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r^2}\left(r\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial \theta^2}\right)\right]_{r=a} = 0 \;; \tag{2.1}$$

b) vanishing transverse reaction force

$$D\left[\frac{\partial}{\partial r}\left(\nabla^2 w\right) + \frac{(1-\nu)}{r^3}\left(r\frac{\partial^3 w}{\partial r \partial \theta^2} - \frac{\partial w}{\partial \theta}\right)\right]_{r=a} = 0 \;; \tag{2.2}$$

c) vanishing stretching force

$$[N_r]_{r=a} = 0 \;; \tag{2.3}$$

2) At the inner edge:

d) vanishing deflexion

$$[w]_{r=b} = 0 \;; \tag{2.4}$$

e) vanishing slope

$$[\partial w/\partial r]_{r=b} = 0 \;; \tag{2.5}$$

f) vanishing radial displacement

$$[u]_{r=b} = 0 \;. \tag{2.6}$$

A distinction must be made between discs which are "built-in" and discs which are "clamped" at $r = b$. By "built-in", we mean that the disc forms an integral part of the shaft or hub to which it is attached. By "clamped" we mean that the disc is a separate piece which is centrally supported over a "clamping circle" by collars or similar constraints to transverse motion. A turbine wheel, typically, is built-in, whereas a sawblade or computer memory disc, typically, is clamped.

For clamped discs, the boundary conditions at $r = b$ differ from those listed above. The slope and deflexion still must vanish, but the clamping constraints do not force the radial displacement to vanish. We therefore acquire another boundary value problem: the elastic deflexion, due to the centrifugal loading, of the portion of the disc within the clamping circle. This is coupled to the boundary value problem we already have by requiring the radial displacement and the radial stretching force to be continuous at the clamping circle.

For a tightly clamped disc, Coulomb friction between the disc and the clamping collars tends to minimize the distinction between a clamped disc and a built-in disc. Bulkeley[24] has recently investigated, in some detail, the effects of clamping on free vibrations of a spinning disc.

III. APPROXIMATION METHODS

Considerable simplification of the disc equations can be achieved by adopting some or all of the following assumptions.

a. Flexural rigidity may be neglected; if the disc behaves as a membrane, $D = 0$. The $D\nabla^4 w$ drops out of equation (1.9), so that we lose some ability to satisfy boundary conditions. We must, in essence, drop one boundary condition at each edge. For the inner edge, the choice is easy: since it is pointless to speak of a built-in membrane, the slope condition does not apply. For the free outer end, setting $D = 0$ loses both condition (2.1) and (2.2), so that a boundary condition must be recovered. By definition, an element of membrane can support only forces which act in its plane. Consequently, we obtain a slope condition at the free edge: the membrane must be parallel to the net force field. In applying this boundary condition we must take into account all forces: body forces, surface forces, D'Alembert forces.

b. If the disc slopes are small, the strains are linear functions of the displacements. The w-dependence drops out of equations (1.10), (1.11), and (1.12), so that

$$\epsilon_r = \partial u/\partial r \ ,$$
$$r\epsilon_\Theta = u + \partial v/\partial\Theta \ ,$$
$$r\gamma_{r\Theta} = r\partial v/\partial r - v + \partial u/\partial\Theta \ . \tag{3.1}$$

c. If the deflexion of the disc is axially symmetric, all $\partial/\partial\Theta$ terms disappear from the various equations. If, in addition, the disc is not submitted to in-plane shear, axisymmetric or otherwise, we may assume $N_{r\Theta} = 0$. In this case, some care must be used in applying compatibility relations, for equation (1.18) will underdetermine the problem: there won't be enough boundary conditions available for a compatibility equation involving second derivatives. Instead, we note that, with $\partial/\partial\Theta$ terms set equal to zero, u can be eliminated from equations (1.10) and (1.11) by a single differentiation:

$$\epsilon_r - \frac{\partial}{\partial r}\ (r\epsilon_\Theta) = \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2. \tag{3.2}$$

With (1.13) and (1.14) we then obtain the equation of compatibility of stress

$$N_r - \frac{\partial}{\partial r}\ (rN_\Theta) - \nu N_\Theta + \nu\frac{\partial}{\partial r}\ (rN_r) - Ed\left(\frac{\partial w}{\partial r}\right)^2. \tag{3.3}$$

d. If the lubricating air can be assumed incompressible, equation (1.20) becomes linear in the pressure (one of those p's in each term is really the lubricant density, proportional to the pressure in an isothermal gas; in an incompressible fluid, the density is constant and cancels out). Thus

$$\frac{\partial}{\partial r}\left(rh^3\ \frac{\partial p}{\partial r}\right) + \frac{\partial}{\partial\phi}\left(h^3\frac{\partial p}{\partial\phi}\right) = 6\mu\Omega r\ \frac{\partial h}{\partial\phi} + 12\mu r\frac{\partial h}{\partial t}\ \ . \tag{3.4}$$

e. If the bearing is but lightly loaded, we may assume

$$h = h_o + w \ ,$$

where h_o is a specified function of r and ϕ, and neglect terms of the second degree or higher in w/h_o. This approach was applied by Langlois[25] to the problem of an inelastic tape passing near a circular cylinder.

IV. STRESS-FUNCTION FORMULATION

The set of equations (1.2), (1.3), (1.9), (1.18) can be expressed more concisely if the stretching forces are eliminated by introducing a stress-function*. In terms of a stress-function Φ, the stretching forces are given by

$$\begin{aligned} N_r &= \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \Theta^2} + \Lambda , \\ N_\Theta &= \frac{\partial^2 \Phi}{\partial r^2} + \Lambda , \\ N_{r\Theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Phi}{\partial \Theta}\right), \end{aligned} \tag{4.1}$$

where Λ is a body force potential defined such that

$$R = -\partial\Lambda/\partial r . \tag{4.2}$$

For the spinning disc,

$$\Lambda = -\rho d\,\Omega^2 r^2 . \tag{4.3}$$

Substituting (4.1) into the compatibility equation (1.9) yields

$$D\nabla^4 w = -2\rho d\frac{\partial^2 w}{\partial t^2} + L(w,\Phi) + \Lambda\nabla^2 w + \frac{\partial w}{\partial r}\frac{\partial \Lambda}{\partial r} + p - p_a , \tag{4.4}$$

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* See Reference 20, p. 418 or Reference 22, p.45-11. Two misprints occur in the latter reference. The second of equations (45.21b) should read

$$-2\nabla^2\nabla^2\Phi = EtL(w,w) - EtL(w_o, w_o)$$

and the expression for $L(w,\Phi)$ should have the term $\frac{1}{r^2}\frac{\partial^2 w}{\partial \Theta^2}$ replaced by $\frac{1}{r}\frac{\partial^2 w}{\partial \Theta^2}$.

where the operator $L(w, \Phi)$ is defined by

$$L(w, \Phi) = \frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \Theta^2}\right) + \frac{\partial^2 \Phi}{\partial r^2}\frac{1}{r}\left(\frac{\partial w}{\partial r} + \frac{1}{r}\frac{\partial^2 w}{\partial \Theta^2}\right)$$
$$- 2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \Theta}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Phi}{\partial \Theta}\right). \qquad (4.5)$$

Substituting (4.1) into (1.18) and using (1.2), (1.3), we obtain

$$-\nabla^4\Phi = EdL(w, w) + (1-\nu)\nabla^2\Lambda + (\nu/r^2)\,\partial^2\Lambda/\partial\Theta^2. \qquad (4.6)$$

The equilibrium equations (1.2) and (1.3) are automatically satisfied by (4.1), provided only that Λ is not a function of Θ. Since for a spinning disc Λ depends only on r, we omit the last term in (4.6):

$$-\nabla^4\Phi = EdL(w, w) + (1-\nu)\nabla^2\Lambda \; . \qquad (4.7)$$

Equations (4.4) and (4.7) form a system equivalent to (1.2), (1.3), (1.9), (1.18). When appropriate boundary conditions are employed, their solution yields two functions, w and Φ. The stretching forces are subsequently obtained from (4.1).

Equations (4.4) and (4.7) govern the motion of a spinning membrane if we set $D = 0$ in (4.4) and modify the boundary conditions appropriately.

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